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D-particle bound states and the D-instanton measure

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Abstract: A connection is made between the Witten index of relevance to threshold bound states of D-particles in the type IIA superstring theory and the measure that enters D-instanton sums for processes dominated by single multiply-charged D-instantons in the type IIB theory.

Keywords: String Duality, Branes in String Theory, M(atrix) Theories.

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Contents

1 Introduction

Since D-particles in ten dimensions are Kaluza–Klein modes of eleven-dimensional supergravity there must be precisely one normalizable D-particle state with charge N [1]. This strongly suggests that there should be one threshold bound state of charge N and mass $Ne^{-\phi}$ in the supersymmetric $SU(N)$ Yang–Mills quantum mechanical system that describes the low energy dynamics of N interacting singly-charged ('fundamental') D-particles [2, 3]. The existence of these bound states is an essential link in the interconnecting duality chains that relate all the string theories and eleven-dimensional supergravity and it is one of the key ingredients in the matrix model [4]. Up to now it has been shown [5, 6] that a suitably defined Witten index, $\mathrm{tr}(-1)^F$, is equal to one in the two fundamental D-particle system. This implies that there is at least one threshold bound state in the two D-particle system. Further evidence for the existence of a bound state when N is prime was given in [7].

The world-line of a charge-n D-particle of the euclidean type IIA string theory winding $\hat{m}$ times around a compact dimension may be identified, via T-duality, with a D-instanton of the compactified IIB theory [8]. The euclidean D-particle action translates into an expression for the action of a D-instanton with instanton number $N = \hat{m}n$ of the form

$$S^N = 2\pi i\rho\hat{m}n\,, \tag{1.1}$$

where $\rho = C^{(0)} + ie^{-\phi^B}$ is the complex scalar field of type IIB supergravity (ϕ^B is the IIB dilaton and $C^{(0)}$ is the Ramond–Ramond ($R \otimes R$) scalar). The spectrum of D-particle states is thereby related directly to the measure of multiply-charged D-instantons. It is

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therefore natural to ask if information concerning the moduli space of D-instantons might prove useful in understanding the D-particle bound states.

Such information is contained in certain terms in the M-theory or type II string theory effective actions that are plausibly given entirely by tree-level, one-loop and non-perturbative D-instanton corrections [8, 9, 10]. These terms include a $f(\rho,\bar{\rho})R^4$ term (where R^4 indicates a particular contraction of four Riemann curvatures), a $f_{16}(\rho,\bar{\rho})\lambda^{16}$ term [11] (where λ is the spin-1/2 fermion in the type IIB theory) and other terms related to these by supersymmetry. The functions $f(\rho,\bar{\rho})$ and $f_{16}(\rho,\bar{\rho})$ are non-holomorphic modular forms of appropriate weight. The sum over the infinite set of D-instantons which emerges from the asymptotic behaviour of f and f_{16} includes a measure factor that should be related to the calculation of the Witten index of the D-particle bound state problem. This connection is the subject of this paper.

2 D-instanton moduli space

Classical D-instantons are solutions of euclidean IIB supergravity [12] which preserve half the euclidean supersymmetry in which all the fields are trivial (in Einstein frame) apart form the complex scalar. The solution describing k D-instantons located at space-time points y_i^μ ($\mu = 0,\ldots,9$ and $i = 1,\ldots,k$) each carrying an integer 'charge' N_i is defined by the dilaton profile

$$e^{\phi(x)-\phi_\infty} = 1 + \sum_{i=1}^{k} \frac{2N_i e^{-\phi_\infty}}{\pi^{3/2}|x-y_i|^8}\,, \tag{2.1}$$

together with $dC^{(0)} = -ide^{-\phi}$. The string coupling constant is identified with the asymptotic value of the dilaton, $g = e^{\phi_\infty}$ and the action for this configuration is given by $S^{\{N_i\}} = \sum_i 2\pi i N_i \rho_0$, where $\rho_0 = \chi + i/g$ is the asymptotic value of ρ.

Sixteen of the thirty-two IIB supersymmetries are conserved in the background of a single charge-N D-instanton while the broken supersymmetries generate fermionic collective coordinates which are the super-partners of the bosonic collective coordinates, y^μ. The integration over these fermionic zero modes induces new local interactions (as with t'Hooft vertices). Multi-instanton solutions have more bosonic and fermionic zero modes and therefore contribute to effective interaction vertices with more derivatives. In the following we will be considering interaction terms that are integrals over sixteen fermionic coordinates, or half the superspace, and are therefore dominated by the single D-instantons. These interactions are expected to be protected by non-renormalization theorems.

As is well known, the classical supergravity solutions describing D-branes are only appropriate for describing physics at large separations. At small D-brane separations the appropriate dynamical degrees of freedom are the open strings connecting them and the low energy excitations of the N D-brane system are associated with the fermionic and bosonic open-string ground states. The D-brane effective world-volume theory is thereby identified with the dimensional reduction of ten-dimensional $U(N)$ supersymmetric Yang-Mills theory to $p+1$ dimensions [13]. The situation with D-instantons is somewhat degenerate since $p = -1$ and the world-volume reduces to a space-time point. In this case the open strings have end-points fixed in all space-time dimensions so they do not describe propagating degrees of freedom. The configuration space of N D-instantons is determined by bosonic and fermionic $U(N)$ matrices A_μ and ψ_α, where μ and α denote $SO(10)$ vector and spinor indices, respectively. The reduction of the supersymmetric Yang–Mills action to a point is given by

$$S = \frac{1}{4}\mathrm{tr}([A_\mu, A_\nu]^2) + \frac{i}{2}\mathrm{tr}(\bar{\psi}\Gamma^\mu[A_\mu, \psi])\,. \tag{2.2}$$

The 'center of mass' degrees of freedom are associated with the element of the Cartan subalgebra of $U(N)$ proportional to the unit matrix and do not appear in the action (2.2). They do, however, enter as shift symmetries in the supersymmetry transformations,

$$\begin{aligned} \delta A_\mu &= i\bar{\eta}\Gamma_\mu\psi\,, \\ \delta\psi &= [A_\mu, A_\nu]\Gamma^{\mu\nu}\eta + \mathbf{1}\,\epsilon\,. \end{aligned} \tag{2.3}$$

The sixteen-dimensional spinor η parameterizes the dimensionally reduced supersymmetry of the $SU(N)$ YM theory and the spinor ϵ acts as a constant shift on the fermions in the 'center of mass' degrees of freedom. Hence the U(1) part of $U(N)$ fermionic fields ψ plays the role of the sixteen fermionic collective coordinates associated with the charge-N D-instanton.

After factoring out the centre of mass coordinates the N D-instanton partition function is given by

$$Z = \int d^{10}y \int d^{16}\epsilon\, Z_{SU(N)}\,, \tag{2.4}$$

where

$$Z_{SU(N)} = \int_{SU(N)} D\psi DA \exp(-S_{SYM}[A,\psi])\,. \tag{2.5}$$

The integration over y^μ and ϵ^A is the integral over the overall bosonic and fermionic collective coordinates of the collection of D-instantons of total charge N. A non-zero value for $Z_{SU(N)}$ can only arise from configurations in which there are no extra fermionic zero modes (in addition to ϵ) which is characteristic of a single multiply-charged D-instanton. The quantity $Z_{SU(N)}$ should therefore be identified with the measure on the space of single charge-N D-instantons. As we shall see, the integral (2.5) is the same as the bulk integral which appears in the calculation of the Witten index in the bound state problem of D-particles. Although this integral has not been evaluated for general N it has been explicitly evaluated in the case $N = 2$ [5, 6]. The sub-integral over configurations in which the elements of the Cartan subalgebra, A^3_μ, ψ^3, are fixed was evaluated in [24] and corresponds to two D-instantons at a fixed separation.

3 D-instanton terms in the effective action

The presence of a D-instanton induced R^4 vertex in IIB string theory was demonstrated in [8]. This, together with the known perturbative string tree-level [14, 15] and one-loop

[16, 17] R^4 terms, motivated the conjecture that the exact R^4 term in IIB superstring theory is given by [8]

$$S_{R^4} = \int d^{10}x\sqrt{g}\rho_2^{1/2} f(\rho,\bar\rho) t_8 t_8 R^4\,, \tag{3.1}$$

where $g^B_{\mu\nu}$ is the ten-dimensional IIB string-frame metric and the function f is a non-holomorphic modular function defined by

$$f(\rho,\bar\rho) = \sum_{(n,m)\neq(0,0)} \frac{\rho_2^{3/2}}{|m+n\rho|^3}\,. \tag{3.2}$$

The large-ρ_2 (small coupling) expansion of f is given by

$$\begin{aligned} \rho_2^{1/2} f(\rho,\bar\rho) &= 2\zeta(3)(\rho_2)^2 + \frac{2\pi^2}{3} + 4\pi^{3/2}\sum_N (N\rho_2)^{1/2}\sum_{N|\hat m}\frac{1}{\hat m^2} \\ &\times\left(e^{2\pi i N\rho}+e^{-2\pi i N\bar\rho}\right)\left(1+\sum_{k=1}^{\infty}(4\pi N\rho_2)^{-k}\frac{\Gamma(k-1/2)}{\Gamma(-k-1/2)k!}\right)\,, \end{aligned} \tag{3.3}$$

where $\sum_{N|n}$ denotes the sum over divisors of N. This expression reproduces precisely the perturbative tree-level and one-loop terms and suggests a perturbative non-renormalization theorem beyond one loop for R^4 and other terms related by supersymmetry [8]. Recently, other arguments have lent support to this [18, 19]. The non-perturbative terms in (3.3) are contributions of D-instantons (and anti-D-instantons) of arbitrary charge N together with an infinite sequence of perturbative fluctuations around each instanton configuration. The expression obtained in [8] is reproduced by writing $N = n\hat m$, for integer $\hat m$. Following T-duality on an euclidean circle $\hat m$ can be reinterpreted as the winding number of a charge-n D-particle world-line as described in the introduction.

Similar remarks apply to other terms related by supersymmetry to the $t_8t_8R^4$ term, such as the sixteen-fermion λ^{16} term (where λ is the complex chiral spin-1/2 fermion of the IIB theory). Since λ carries charge 3/2 under the $U(1)$ R-symmetry of the IIB theory it transforms under $SL(2,Z)$ with holomorphic and anti-holomorphic weights $(3/4,-3/4)$ so that λ^{16} has weight $(12,-12)$. An argument was given in [11] that the λ^{16} contribution to the IIB action has the form

$$\int d^{10}x\sqrt{g}\rho_2^{1/2}\hat K f_{16}(\rho,\bar\rho)\lambda^{16}\,, \tag{3.4}$$

where $\hat K$ is a kinematic coefficient and f_{16} is a specific non-holomorphic modular form that transforms with holomorphic and anti-holomorphic weights $(-12,12)$ under $SL(2,Z)$. More generally, supersymmetry relates a large number of terms of the same dimension which are characterized by their $U(1)$ charge, which is an even integer, $2r$. The functions of ρ and $\bar\rho$ which multiply these terms in the action are generalized non-holomorphic modular forms, $F_{(r,-r)}(\rho,\bar\rho)$, with weights $(r,-r)$. These are related to $f(\rho,\bar\rho)$ (3.2) by

$$F_{(r,-r)}(\rho,\bar\rho) = \rho_2^r D^r f(\rho,\bar\rho)\,, \tag{3.5}$$

where D is the (nonholomorphic) covariant derivative that maps generalized modular forms of weight (l,l') to forms of weight $(l+2,l')$. The action of D on the generalized holomorphic forms $F_{(l,-l)}$ is defined by

$$DF_{(l,-l)} \equiv i\left(\frac{\partial}{\partial\rho} + \frac{l}{\rho-\bar\rho}\right)F_{(l,-l)}\,. \tag{3.6}$$

The modular form that enters into the λ^{16} interaction is given by $f_{\lambda^{16}} = F_{(12,-12)}(\rho,\bar\rho)$ [11]. The leading N-instanton contributions to the large-ρ_2 expansion of the scalar function $F_{(r,-r)}$ can easily be determined from (3.3) and (3.5) to be given by

$$F_{(r,-r)} \sim C_k \sum_N (N\rho_2)^{r+1/2} e^{2\pi i N\rho} \sum_{N|\hat m} \frac{1}{\hat m^2} + \cdots\,, \tag{3.7}$$

where C_k is a constant and $\cdots$ represents perturbative corrections to the instanton contribution that are suppressed by powers of the inverse instanton action, $1/(N\rho_2)$. We wish to focus on the measure factor,

$$\sum_{N|\hat m} \frac{1}{\hat m^2}\,, \tag{3.8}$$

which is common to all processes (all k) and should therefore be identified, up to an overall constant, with the measure of singly-charged D-instantons, $Z_{SU(N)}$.

4 D-particles and D-instantons

The Witten index for the system of N D-particles was defined in [5, 6] by

$$I^{(N)} = \int dx \lim_{\beta\to\infty} \mathrm{tr}(-1)^F e^{-\beta H}(x,x) \tag{4.1}$$

$$= \lim_{R\to\infty}\lim_{\beta\to 0}\left\{\int_{|x|<R} dx\ \mathrm{tr}(-1)^F e^{-\beta H} + \frac{1}{2}\int_{|x|=R} dx \int_\beta^\infty d\beta'\ \mathrm{tr}e_n(-1)^F Q e^{-\beta' H}\right\} \tag{4.2}$$

where the first term defines a bulk contribution $I^{(N)}_{bulk}$ (which in general is not an integer) and the second term a deficit contribution $I^{(N)}_{def}$ (which corrects the bulk term). The separation of the integer $I^{(N)}$ into these two parts is dependent on the order of limits. We shall always consider the infinite volume limit to be taken before the limit $\beta\to 0$.

The trace in (4.1) is defined over gauge invariant states of the Hilbert space, which may be written as a trace over all states if a projector onto gauge invariant states is inserted. For the bulk term this gives

$$I^{(N)}_{bulk} = \lim_{\beta\to 0}\int_{SU(N)} d\eta \int dx\ \mathrm{tr}(-1)^F e^{i\eta C} e^{-\beta H}(x,x)\,. \tag{4.3}$$

Using the heat kernel approximation for the propagator in the hamiltonian H, which is valid in the limit $\beta\to 0$, it was observed in [5] and [6] that the exponent of the bulk part of the index can be written in a $SO(10)$ invariant form. This is possible because the parameter η turns into a tenth bosonic coordinate A_0. The resulting integrals are by closer inspection exactly equivalent to the $SU(N)$ integral of the D-instanton collective coordinates in $Z_{SU(N)}$ (2.5). This is simply the statement of T-duality for the bulk term, since in (4.3) the radius β of the euclidean circle shrinks to zero size. Note that the insertion of $(-1)^F$ enforces supersymmetric boundary conditions on the fermionic fields.

The conjectured form for the measure, $Z_{SU(N)}$ given by (3.8), therefore suggests that the value of the bulk integral is given, for arbitrary N, by

$$I_{bulk}^{(N)} = \sum_{N|\hat{m}} \frac{1}{\hat{m}^2}, \tag{4.4}$$

where an overall constant has been fixed by noting that $I_{bulk}^{(1)} = 1$. Note that for $N = 2$ the value of $I_{bulk}^{(2)} = 5/4$ in (4.4) agrees with the explicit calculation of [5, 6]. When N is prime the conjectured expression for the bulk integral is

$$I_{bulk}^{(N)} = \left(1 + \frac{1}{N^2}\right) . \tag{4.5}$$

5 The deficit term

The appearance of a deficit term is related to the presence of boundary terms produced by an integration by parts that is necessary in moving Q around in the trace in (4.1). The only sizeable contribution to the boundary term comes from propagation along the flat directions of the potential. A heuristic evaluation of the deficit term in the $SU(2)$ case was given by Yi [5] who argued that it comes from the region in which the two D-particles are separated and the $SU(2)$ is broken to $U(1)$. He suggested that the particles might behave as identical free particles in this region, with moduli space R^9/Z_2. Since the Witten index vanishes for free particles the deficit term, $I_{def}^{(2)}$, must cancel the bulk term for the free system, $I_{0\,bulk}^{(2)}$. But this free bulk term is easily evaluated and is equal to $1/4$. A more precise discussion of the two D-particle system based on an analysis of the integration over the massive modes proves that the deficit term is indeed determined by free propagation on R^9/Z_2 [6]. In the following we shall assume that this prescription generalizes to $N > 2$ D-particles so that, for *prime* values of N,

$$I_{def}^{(N)} = -I_{0\,bulk}^{(N)} , \tag{5.1}$$

where $I_{0\,bulk}^{(N)}$ is the bulk index for N identical free particles moving on $R^{9(N-1)}/S_N$. For *non-prime* values, $N = \hat{m}n$, the generalization will take into account the regions of moduli space of $\hat{m}$ free charge-n particles on $R^{9(\hat{m}-1)}/S_{\hat{m}}$.

In the $N = 2$ case considered in [5, 6] the only configuration that contributed to the trace over the free two-particle states was one with an odd permutation of the two particles. If the trace is expressed as a functional integral this includes only configurations in which the two D-particles are described by a single euclidean world-line that winds twice around the compact β direction. In order to generalize this to arbitrary N we need to consider the action of S_N, which is the Weyl group of $SU(N)$. This can be parameterized by matrices M_{ab} acting on the positions of the N particles modded out by the overall translation invariance, X_a^i ($i = 1, \ldots, 9$), and the fermions ψ_a^α. The index $a = 1, .., N-1$ labels the different $U(1)$'s in the Cartan subalgebra. The action of an element in S_N is given by

$$X_a^i \to M_a^b X_b^i \,, \qquad \psi_a^\alpha \to M_a^b \psi_b^\alpha \,, \tag{5.2}$$

where the vector index runs over $i = 1, \ldots, 9$ and the spinor index $\alpha = 1, \ldots, 16$.

The fermion fields ψ satisfy the following anticommutation relations,

$$\{\psi_a^\alpha, \psi_b^\beta\} = \delta_{ab}\delta^{\alpha\beta} \,. \tag{5.3}$$

It is convenient to build up the fermionic Hilbert space by defining fermionic creation and anihilation operators $\hat{\psi}_a^\alpha, \hat{\psi}_a^{\dagger\alpha}$ by

$$\hat{\psi}_a^\alpha = \frac{1}{\sqrt{2}}\left(\psi_a^{2\alpha-1} + i\psi_a^{2\alpha}\right), \quad \hat{\psi}_a^{\dagger\alpha} = \frac{1}{\sqrt{2}}\left(\psi_a^{2\alpha-1} - i\psi_a^{2\alpha}\right), \quad \alpha = 1, \ldots, 8 \,, \tag{5.4}$$

which satisfy $\{\hat{\psi}_a^{\dagger\alpha}, \hat{\psi}_b^\beta\} = \delta_{ab}\delta^{\alpha\beta}$. A general wave function $\mid \Psi\rangle$ can be expanded as

$$\mid \Psi\rangle = \left(\Psi^{(0)}(X_a) + \Psi_{a_1}^{(1)}(X_a)\hat{\psi}_{a_1}^\dagger + \frac{1}{2}\Psi_{a_1 a_2}^{(2)}(X_a)\hat{\psi}_{a_1}^\dagger\hat{\psi}_{a_2}^\dagger + \cdots\right) \mid 0\rangle \,, \tag{5.5}$$

where the vector and spinor indices are suppressed and the highest term has $8 \times N$ fermionic creation operators acting on the Fock space vacuum.

The expression for the bulk contribution to the Witten index for N free D-particles (with N a prime) is given by

$$I_{0\,bulk}^{(N)} = \lim_{\beta\to 0} \mathrm{tr}(-1)^F e^{-\beta H} \,. \tag{5.6}$$

The trace is taken over gauge invariant states in the Hilbert space and hence one has to insert a projector on states which are invariant under the Weyl permutation group,

$$\mathcal{P} = \frac{1}{N!}\sum_{\pi\in S_N} M_\pi \,. \tag{5.7}$$

The matrix M_π representing $\pi \in S_N$ act as in (5.2) on the coordinates X_a and the fermions $\hat{\psi}_a$ in the wave function Ψ given in (5.5). The bulk index for the free theory is therefore given by

$$I_{0\,bulk}^{(N)} = \lim_{\beta\to 0} < \Psi|(-1)^F e^{-\beta H}\mathcal{P}|\Psi > \,. \tag{5.8}$$

The factor of $(-1)^F$ counts bosons (even number of $\hat{\psi}$) with $+1$ and fermions (odd number of $\hat{\psi}$) with -1. Because the fermions transform under the symmetric group matrices M, the introduction of the projector $\mathcal{P}$ can give a non-vanishing contribution.

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The action of M_π on $\hat\psi_a$ and X_a in (5.8) factorizes into a trace over the Hilbert space build from the fermionic creation operators and a bosonic gaussian integral coming from the heat kernel approximation for the free propagator in the limit $\beta \to 0$ [5]:

$$I^{(N)}_{0\,bulk} = \lim_{\beta\to 0} \frac{1}{N!} \sum_{\pi\in S_N} \mathrm{tr}_\psi \left((-1)^F M_\pi\right) \int \prod_{i=1}^{9} \prod_{a=1}^{N-1} dX^i_a \, \frac{e^{-(X-M_\pi X)^2/2\beta}}{(2\pi\beta)^{(n-1)9/2}} \,. \tag{5.9}$$

For all values of α the fermionic trace is given by

$$\begin{aligned} \mathrm{tr}_\psi \left((-1)^F M_\pi\right) &= \langle 0 \mid 0\rangle - \sum_a \langle 0 \mid \hat\psi_a M_{\pi\, ab} \hat\psi^\dagger_b \mid 0\rangle \\ &\quad + \frac{1}{2} \sum_{a,b} \langle 0 \mid \hat\psi_a \hat\psi_b M_{\pi\, ac} M_{\pi\, bd} \hat\psi^\dagger_c \hat\psi^\dagger_d \mid 0\rangle + \cdots \\ &= 1 - \mathrm{tr}(M_\pi) + \frac{1}{2}\mathrm{tr}(M_\pi^2) - \frac{1}{2}\mathrm{tr}(M_\pi)^2 + \cdots \\ &= \det(1 - M_\pi)\,, \end{aligned} \tag{5.10}$$

so that the trace over all eight fermion components gives a factor of $\det(1 - M_\pi)^8$. The gaussian integration over the coordinates X^i_a similarly gives a factor of $\det(1 - M_\pi)^{-9}$

The determinant is easily evaluated by using an explicit representation for the matrices M_π. Recall that these represent the action of S_N on the elements of the Cartan subalgebra. The roots of $SU(N)$ are the vectors in R^N,

$$e_i - e_j, \qquad i \neq j; \quad i, j = 1, \ldots, N\,, \tag{5.11}$$

where e_i the ith unit vector of R^N. All the roots lie in a $(N-1)$-dimensional subspace $\mathcal{R}_\omega$ orthogonal to the vector $\omega = e_1 + e_2 + \cdots + e_N = (1, 1, \cdots, 1)$ and an element $\pi \in S_N$ acts as a permutation, $\pi : e_i \to e_{\pi(i)}$.

Two clases of permutations need to be distinguished:

(a) *Cyclic permutations*

These can be represented up to conjugation by the $N \times N$ matrix

$$M_c = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{5.12}$$

which has eigenvalues

$$\lambda_k = e^{\frac{2\pi i k}{N}}, \quad k = 0, 1, \ldots, N-1\,. \tag{5.13}$$

The matrix M_c does not leave any of the roots (5.11) invariant and ω is the unique eigenvector with eigenvalue $\lambda_0 = 1$. We are interested in evaluating $\det(1 - M_\pi) = \det'(1 - M_c)$, where the prime indicates the omission of the zero eigenvalue. This is the determinant in the space orthogonal to ω and is given by the product of non-zero eigenvalues,

$$\det{}'(1 - M_c) = \prod_{k=1}^{N-1} (1 - \lambda_k) = 2^{N-1} \prod_{k=1}^{N-1} \sin\frac{\pi k}{N} = N\,. \tag{5.14}$$

(b) *Non-cyclic permutations*

These can be represented by

$$M_{nc} = \begin{pmatrix} M_1 & 0 & 0 & \cdots & 0 \\ 0 & M_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & M_p \end{pmatrix}, \tag{5.15}$$

where M_1, M_2, $\cdots$, M_p, represent cyclic permutations of subsets of elements and can be written in the form of (5.12). The matrix M_{nc} has $p > 1$ unit eigenvalues so there are $p-1$ eigenvectors $w_j \in R_\omega$ for which $(1 - M_{nc})w_j = 0$ $(j = 1, \ldots, p-1)$. As a result, some elements of the Cartan subalgebra are left invariant by M_π and

$$\det(1 - M_\pi) = \det{}'(1 - M_{nc}) = 0\,, \tag{5.16}$$

where the prime again indicates the omission of the zero eigenvalue associated with ω.

To be more precise, a zero eigenvalue of the bosonic determinant $\det(1 - M_{nc})$ should be interpreted as the inverse volume, R^{-1} in the limit $R \to \infty$, whereas the fermionic determinant vanishes identically. This means that only the cyclic permutations contribute to the index and the expression (5.9) reduces to

$$I^{(N)}_{0\,bulk} = \sum_{c \in S_N} \frac{1}{N!} (\det{}'(1 - M_c))^{-1}\,. \tag{5.17}$$

The non-vanishing determinant $\det'(1 - M_c) = N$ simply counts the winding number of a fundamental D-particle world-line and the result is the obvious generalization of the $N = 2$ case. The non-cyclic permutations do not contribute to the index but have the interpretation of p disconnected D-particle world-lines winding around the euclidean circle. This is a multi D-instanton configuration and the vanishing of the determinant is due to the appearance of extra fermionic zero modes. Since each of the $(N-1)!$ cyclic elements of S_N gives the same contribution to (5.17) the result is

$$I^{(N)}_{0\,bulk} = \frac{1}{N^2} = -I^{(N)}_{def}\,. \tag{5.18}$$

Combining this with our earlier ansatz that $I^{(N)}_{bulk} = Z_{SU(N)} = 1 + 1/N^2$ leads to the conclusion that

$$I^{(N)} = I^{(N)}_{bulk} + I^{(N)}_{def} = 1\,, \tag{5.19}$$

for prime values of N. This is consistent with the presence of at least one bound state for every prime value of N.

Now consider the non-prime charge sector in which $N = \hat{m}n$ with $\hat{m}, n > 1$. It is useful to recall the interpretation of the D-instanton as a wrapped euclidean D-particle world-line. The contributions to the instanton measure labelled by n were associated with the world-line of a charge-n D-particle winding $\hat{m}$ times. In the case when N is prime the only contribution to the deficit term comes from $n = 1$, $\hat{m} = N$, which corresponds to the winding of a charge-$\hat{m}$ D-particle world-line. The other possibility, $n = N$, $\hat{m} = 1$, is the contribution to the Witten index of the threshold bound state. It is therefore very compelling to *assume* that every threshold bound state of charge n contributes to the deficit term in the same way as the fundamental ($n = 1$) D-particle. This is a generalization of the property proved in the $N = 2$ case by [6] that the deficit term is determined by free particle dynamics. It means that the space of free-particle states that enters in the calculation of of $I_{0\,bulk}$ should be enlarged to include the infinite tower of charge-n threshold bound states. In that case a cyclic permutation of $\hat{m}$ identical charge-n particles contributes a term

$$I_{0\,bulk}^{(\hat{m})} = \frac{1}{\hat{m}^2}\,, \tag{5.20}$$

which leads to a total deficit in the charge-N sector of

$$I_{def}^{(N)} = -\sum_{\substack{N|\hat{m}\\ \hat{m}>1}} I_{0\,bulk}^{(\hat{m})} = -\sum_{\substack{N|\hat{m}\\ \hat{m}>1}} \frac{1}{\hat{m}^2}\,, \tag{5.21}$$

consistent with the Witten index $I^{(N)} = 1$ for all integer N.

The success of these assumptions points to the systematics that needs to emerge from a more precise treatment of the integration over the boundary term when N is not prime. This suggests that there are regions of moduli space in which the non-abelian integration leads to $\hat{m}$ charge-n threshold bound states which behave as free particles.

6 Conclusions

This paper has made a connection between the measure that enters the charge-N D-instanton corrections to the IIB effective action and the bulk contribution to the Witten index of D-particle quantum mechanics which is related to the existence of threshold charge-N bound states of D-particles. The tight constraints on the D-instanton action implied by the $SL(2,Z)$ duality of IIB string theory lead to a conjectured expression for the measure in the sum over single charge-N D-instantons. This should encode the value of the the complicated $SU(N)$ integrals that need to be evaluated if the measure is calculated directly from (2.5). Related issues arise in the considerations of [23] concerning D-string instantons in $d = 8$ type I string theories.

This is reminiscent of the Seiberg-Witten arguments concerning the solution of $N = 2$ supersymmetric Yang–Mills [20]. In that case duality constrained the form of the measure of the instanton corrections to the prepotential without explicit evaluation of the complicated integrals over the ADHM-moduli space [21]. In this regard it is also interesting to note that the connection between the non-perturbative D-instanton corrections to the effective action in ten dimensions and a regularized one loop term in eleven dimensional M-theory [10] also has an analogy in $N = 2$ supersymmetric Yang–Mills, where the instanton-corrected prepotential can be related to a one-loop term in five-dimensional supersymmetric Yang–Mills theory [22].

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